Mapping class groups
Problem sheet 2

Lent 2021

1. What is the maximal cardinality of a pairwise disjoint, pairwise non-isotopic set of essential simple closed curves on a closed surface S?

2. Exhibit $\phi \in \mathrm{Mod}(S_g)$ of order g.

3. Let S be the closed, orientable surface obtained from a decagon P_{10} by identifying opposite sides in pairs.

   (a) What is the genus of S?

   (b) Let ϕ be the rotation of P_{10} by π. Show that ϕ induces a self-homeomorphism of S. What is its order in $\mathrm{Mod}(S)$?

4. Let γ be an element of a group G. The corresponding *inner automorphism* of G is defined by

$$i_\gamma : g \mapsto \gamma g \gamma^{-1} \ .$$

   Let $\mathrm{Inn}(G)$ denote the image of the map $G \to \mathrm{Aut}(G)$ sending $\gamma \mapsto i_\gamma$.

   (a) Prove that $\gamma \mapsto i_\gamma$ is a homomorphism and that $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$. What is the kernel of the homomorphism $G \to \mathrm{Inn}(G)$?

   (b) The quotient $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the *outer automorphism group* of G. Exhibit a homomorphism $\mathrm{Mod}(S) \to \mathrm{Out}(\pi_1 S)$.

   (c) Is the homomorphism $\mathrm{Mod}(S) \to \mathrm{Out}(\pi_1 S)$ surjective?

5. Let S be closed and hyperbolic, and let $\phi : S \to S$ be an isometry isotopic to the identity. Show that ϕ is equal to the identity.

6. Let G be any finite group.

   (a) Show that there is a surjection $\pi_1 S_g \to G$, for some g.

   (b) Show that G is a subgroup of $\mathrm{Mod}(S_h)$, for some h.

7. Let S be closed and hyperbolic. Let $\phi \in \mathrm{Homeo}^+(S)$ and suppose that, for every simple closed curve α, $\phi \circ \alpha$ is homotopic to α. Show that ϕ is isotopic to the identity.

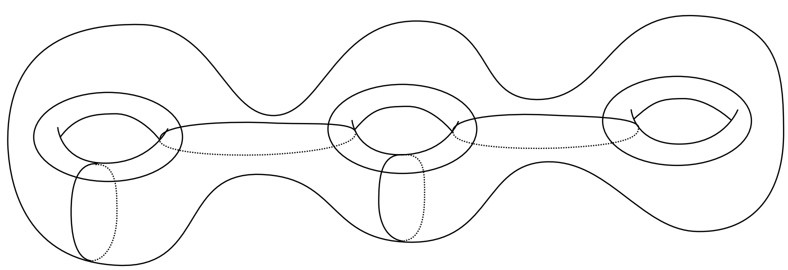
8. Let α, β be a pair of simple closed curves on a surface S, such that $i(\alpha, \beta) = 1$.

   (a) Prove that $T_\alpha T_\beta(\alpha) = \beta$.

   (b) Prove the *braid relation*: $T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta$.

9. Let $S = S_{1,0,1}$, the torus with one boundary component. Let α, β be a standard pair of simple closed curves on S such that $i(\alpha, \beta) = 1$, and let γ be the boundary curve. Prove that $T_\gamma = (T_\alpha T_\beta)^6$.

10. Let S be the closed surface of genus $g \geq 3$. Prove that the centre of $\mathrm{Mod}(S)$ is trivial. [*Hint: consider a suitable collection of simple closed curves on S. For instance, when $g = 3$, consider the curves in the following picture.*]



11. The group $SL_2(\mathbb{Z})$ has a presentation

$$\langle a, b \mid aba = bab\,,\ (ab)^6 = 1 \rangle$$

where a and b correspond to the elementary matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

respectively. Prove that

$$\mathrm{Mod}(S_{1,0,1}) \cong \langle a, b \mid aba = bab \rangle\,.$$